The Product and Quotient Rules

u, v – functions

Q: If we know u' and v' can we calculate $(uv)'$ or $\left(\frac{u}{v}\right)'$?

equality of functions

<u>Guess</u> : $(uv)' = u' \cdot v'$

$$\left(\frac{u}{v}\right)' = \frac{u'}{v'}$$

Lets check with an example:

$$u(x) = x^2, \; v(x) = x \Rightarrow \begin{cases} (uv)(x) = x^3 & \quad \overset{u(x)\,v(x)}{/\!/} \\ \left(\frac{u}{v}\right)(x) = x & \quad \overset{\frac{u(x)}{v(x)}}{=} \end{cases}$$

$$\Rightarrow u'(x) = 2x, \; v'(x) = 1, \; (uv)'(x) = 3x^2$$

$$\left(\frac{u}{v}\right)'(x) = 1$$

not same function

$$(uv)'(x) = 3x^2 \neq 2x \cdot 1 = 2x = u'(x)\,v'(x)$$

$$\left(\frac{u}{v}\right)'(x) = 1 \neq \frac{2x}{1} = 2x = \frac{u'(x)}{v'(x)}$$

So our naive guess is incorrect. Interesting!

<u>Product Rule</u> : $(uv)' = u'v + u \cdot v'$

$$\left(\frac{d(uv)}{dx} = \frac{du}{dx} v + u \frac{dv}{dx} \right)$$

Quotient Rule : $\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$

$$\left(\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{\frac{du}{dx} v - u \frac{dv}{dx}}{(v)^2} \right)$$

Remark There are strange and unexpected. Life is like that sometimes.

Example 1/ $u(x) = x^2$, $v(x) = x$

$u'(x) v(x) + u(x) v'(x) = 2x \cdot x + x^2 \cdot 1$

$= 3x^2 = (uv)'(x)$

$$\frac{u'(x) v(x) - u(x) v'(x)}{(v(x))^2} = \frac{2x \cdot x - x^2 \cdot 1}{x^2} = 1 = \left(\frac{u}{v}\right)'(x)$$

2/ $f(x) = (\sqrt{x} - x)(x^{3/2} - 1) \Rightarrow f'(x) = ?$

Method 1 (Multiply out first)

$f(x) = x^2 - x^{5/2} + x - x^{\frac{1}{2}}$

$\Rightarrow f'(x) = 2x - \frac{5}{2} x^{3/2} + 1 - \frac{1}{2} x^{-\frac{1}{2}}$

Method 2 (Use product Rule)

$u(x) = (\sqrt{x} - x) \Rightarrow \quad u'(x) = \frac{1}{2} x^{-\frac{1}{2}} - 1$

$$v(x) = \left(x^{3/2} - 1\right) \Rightarrow \quad v'(x) = \frac{3}{2} x^{1/2}$$

$$\Rightarrow \quad f'(x) = \left(\frac{1}{2} x^{-\frac{1}{2}} - 1\right)\left(x^{3/2} - 1\right) + \left(x^{\frac{1}{2}} - x\right) \frac{3}{2} x^{1/2}$$

$$= \frac{1}{2} x - \frac{1}{2} x^{-\frac{1}{2}} - x^{3/2} + 1 + \frac{3}{2} x - \frac{3}{2} x^{3/2}$$

$$= 2x - \frac{5}{2} x^{3/2} + 1 - \frac{1}{2} x^{-\frac{1}{2}}$$

3/ $f(x) = \dfrac{x^3 + 6x}{1 - x} \Rightarrow f'(x) = ?$

$$u(x) = x^3 + 6x \Rightarrow u'(x) = 3x^2 + 6$$

$$v(x) = 1 - x \Rightarrow V(x) = -1$$

$$\Rightarrow \quad \frac{u'(x)\,v(x) - u(x)\,v'(x)}{(v(x))^2} = \frac{(3x^2+6)(1-x) - (x^3+6x)(-1)}{(1-x)^2}$$

$$\Rightarrow \quad f'(x) = \frac{3x^2 - 3x^3 + 6 \cancel{- 6x} + x^3 \cancel{+ 6x}}{(1-x)^2}$$

$$= \frac{3x^2 - 2x^3 + 6}{(1-x)^2}$$

4/ $C(x)$ = Cost of producing x units

$$\overline{C}(x) = \frac{C(x)}{x}$$

Marginal Average Cost $= \dfrac{d}{dx}\left(\overline{C}(x)\right) = \overline{C}'(x)$

Find the marginal average cost function if

$$C(x) = \frac{4x+50}{x+2}.$$

$$\Rightarrow \quad \bar{C}(x) = \frac{4x+50}{x^2+2x}$$

$u(x) = 4x+50 \quad \Rightarrow \quad u'(x) = 4$

$v(x) = x^2+2x \quad \Rightarrow \quad v'(x) = 2x+2$

$$\Rightarrow \quad \bar{C}'(x) = \frac{4(x^2+2x) - (4x+50)(2x+2)}{(x^2+2x)^2}$$

$$= \frac{4x^2+\cancel{8x} - 8x^2 - \cancel{8x} + 100x + 100}{(x^2+2x)^2}$$

$$= \frac{100x - 4x^2 + 100}{(x^2+2x)^2}$$